* since we know how to graph polynomials of higher degree when they are factored. This lesson is on how to get them into factor, linear factors (or prime) so we can graph

5.5- Real Zeros of a Polynomial Function

Remainder Theorem

Let $f(x)$ be a polynomial.

If $f(x)$ is divided by $(x-c)$, then the remainder is $f(c)$

quotient divisor remainder

$f(x) = q(x)g(x) + r(x)$

Find the remainder if $f(x) = x^3 + 3x^2 + 2x - 1$ is divided by $x+2$

$f(-2) = (-2)^3 + 3(-2)^2 + 2(-2) - 1$

$= -8 + 12 - 4 - 1 = \boxed{-1}$

remainder is -1

Factor Theorem

Let $f(x)$ be a polynomial.

$(x-c)$ is a factor of $f(x)$ IFF $f(c) =$ 0

Therefore c is a real zero of the polynomial.

Use the Factor Theorem to determine whether the function $f(x) = -2x^3 - x^2 + 4x + 3$ has the factor

(a) $x+1$ (b) $x-1$

a) $f(-1) = -2(-1)^3 - (-1)^2 + 4(-1) + 3 =$

$= 2 - 1 - 4 + 3 = \boxed{0}$ Thus $(x+1)$ is a factor of $f(x)$ & $x=-1$ is a real zero

b) $f(1) = -2(1)^3 - (1)^2 + 4(1) + 3 = -2 - 1 + 4 + 3 = \boxed{4}$

Thus $(x-1)$ is not a factor so $x=1$ is not a real zero

Review of Synthetic Division:

** You can only use this when:

you're dividing a polynomial by a binomial of degree 1 & a leading coefficient of 1

Use Synthetic Division to the following function $f(x) = -2x^3 - x^2 + 4x + 3$

(a) $x+1$ (b) $x-1$

a)
$$\begin{array}{r|rrrr} -1 & -2 & -1 & 4 & 3 \\ & \downarrow & 2 & -1 & -3 \\ \hline & -2 & 1 & 3 & \boxed{0} \end{array}$$

since remainder is zero $(x+1)$ is a factor so $(x+1)(-2x^2 + x + 3) = f(x)$

b)
$$\begin{array}{r|rrrr} 1 & -2 & -1 & 4 & 3 \\ & \downarrow & -2 & -3 & 1 \\ \hline & -2 & -3 & 1 & \boxed{4} \end{array}$$

Thus $(x-1)$ is not a factor of $f(x)$

The choice is yours, you can either use the Factor Theorem to see if you need to actually work it out, OR you can Synthetic Division and see if it works or not.

Theorem: Number of real zeros

A polynomial Can't have more real zeros than it's degree

Rational Zeros Theorem:

The definition is confusing... so in summary:

To find the Potential Rational zeros take all of the positive and negative factors of p (aka the Constant at the end) and all of the positive and negative factors of q (aka the Leading coefficient)

The P.R.Z. are all combinations of: $\frac{p}{q}$

List the potential rational zeros of $f(x) = 3x^3 + 8x^2 - 7x - 12$ (q = 3, p = 12)

p factors: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$

q factors: $\pm1, \pm3$

$\frac{p}{q}$: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

* To Test points (Factor theorem, or synthetic Division)

to see if $(x+1)$ is a factor

$$\begin{array}{r|rrrr} -1 & 3 & 8 & -7 & -12 \\ & \downarrow & -3 & -5 & 12 \\ \hline & 3 & 5 & -12 & \boxed{0} \end{array}$$

Thus $(x+1)(3x^2 + 5x - 12)$

since this is a trinomial we can factor

$\frac{(x+1)(3x+9)(3x-4)}{3}$ → $3(x-3)$

$\boxed{(x+1)(x-3)(3x-4)}$

Steps for finding Real Zeros

Step 1: Determine the maximum number of zeros

Step2: a) Use the P/q test to see what the possible zeros are.

b) Use Synthetic division, Factor theorem, long division to test each possible zero. Once a Zero is found repeat step 2 on depressed equation.

Find the real zeros of

$f(x) = 2x^4 + 13x^3 + 29x^2 + 27x + 9$.

Write f in factored form. ① at most 4 zeros

② p: ±1, ±3, ±9 P/q: ±1, ±1/2, ±3, ±3/2, ±9, ±9/2

q: ±1, ±2

test (x+1)

$$\begin{array}{r|rrrrr} -1 & 2 & 13 & 29 & 27 & 9 \\ & \downarrow & -2 & -11 & -18 & -9 \\ \hline & 2 & 11 & 18 & 9 & \boxed{0} \end{array}$$

since remainder is zero (x+1) is a factor

$(x+1)(2x^3+11x^2+18x+9)$

Now lets try (x-1) from reduced

$$\begin{array}{r|rrrr} 1 & 2 & 11 & 18 & 9 \\ & & 2 & 13 & 31 \\ \hline & 2 & 13 & 31 & \boxed{40} \end{array}$$

(x-1) is not a factor

test (x+3)

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & \downarrow & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & \boxed{0} \end{array}$$

thus (x+3) is a factor

$(x+1)(x+3)(2x^2+5x+3)$

Since its quadratic I can factor

$(x+1)(x+3)\frac{(2x+2)(2x+3)}{2}$

$(x+1)(x+3)\,\cancel{2}(x+1)(2x+3)$

$(x+1)(x+3)(2x+3)$ zeros at $x=-1,-3,-\frac{3}{2}$

Theorem:

Every polynomial function with real coefficients can be uniquely factored into a product of linear factors and/or irreducible prime (quadratic) factors.

Theorem:

A polynomial function with real coefficients of odd degree has at least one real zero.

Can you explain why??

Because it has cubic behavior thus the end behavior would be ↗ or ↘ which at least at one point it must cross the x axis

What is a **bound**? (pg. 381)

$$-M \leq \text{any real zero of } f \leq M$$

Bounds on Zeros:

f is a polynomial function whose leading coefficient is 1. A bound, M, on the real zeros of f is the smaller of the two numbers:

$$\boxed{\text{Max}\{1, |a_0| + |a_1| + \cdots + |a_{n-1}|\}}$$

or

$$\boxed{1 + \text{Max}\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}}$$

Examples of how to find the **bound**:

$$f(x) = x^5 + 3x^3 - 9x^2 + 5$$

Since L.C is already 1 I don't need to factor out the Leading coefficient.

$\max\{1, \underbrace{|a_0|+|a_1|+\cdots+|a_{n-1}|}\}$ add all coefficient except for the Leading Coefficient

$\max = \{1, |5| + |-9| + |3|\} = \{1, 17\} = \boxed{17}$

or

$1+\max\{|a_0|, |a_1|, \cdots, |a_{n-1}|\}$ Pick the greatest absolute value of one coefficent & add 1 to it

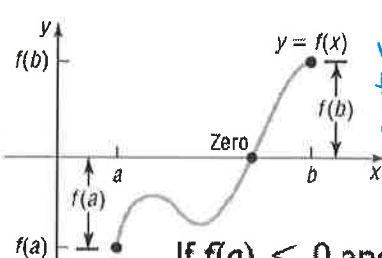
$= 1 + \max\{|5|, \cancel{|0|}, |-9|, |3|\}$ → biggest is $|-9| = 9$

$= 1 + |-9| = 1 + 9 = \boxed{10}$ the smaller number between 10 & 17 is 10 Thus the every real zero of f lies between the bound -10 to 10

The bound on f(x) is 10

Intermediate Value Theorem:

f is a polynomial function. If $a < b$ and if $f(a)$ and $f(b)$ are of opposite signs. Then there exists at least one real zero of f between a & b.



Just like why we know that a polynomial of odd degree must have at least one real zero

If $f(a) < 0$ and $f(b) > 0$ and if f is continuous, there is at least one zero between a and b.

5.5 extra Examples

* Finding the Real zeros

$f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$

*there are at **most** 5 real zeros

* $\frac{p}{q}$: Factors of p: $\pm1, \pm2, \pm4, \pm8, \pm16$
  Factors of q: ±1

$\frac{p}{q} = \pm1, \pm2, \pm4, \pm8, \pm16$

- lets test $(x+1)$

$$\begin{array}{r|rrrrrr} -1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & \downarrow & -1 & 6 & -18 & 42 & -74 \\ \hline & 1 & -6 & 18 & -42 & 74 & \boxed{-90} \end{array}$$

→ not a zero

- lets test $(x-1)$

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & \downarrow & 1 & -4 & 8 & -16 & 16 \\ \hline & 1 & -4 & 8 & -16 & 16 & \boxed{0} \end{array}$$

& $x=1$ is a zero

$(x-1)(x^4-4x^3+8x^2-16x+16)$ → we still need to find more zeros

- lets test $(x-2)$
  use reduced f(x)

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & \downarrow & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & \boxed{0} \end{array}$$

→ $x=2$ is a zero

$(x-1)(x-2)(x^3-2x^2+4x-8)$ still need more

- lets see if it will factor by grouping since there's 4 term

$(x^3-2x^2)+(4x-8)$

GCF is x^2 GCF is 4

$x^2(x-2)+4(x-2)$

Take GCF again $(x-2)(x^2+4)$

Thus

$(x-1)(x-2)(x-2)(x^2+4)$ → prime

so $\boxed{(x-1)(x-2)^2(x^2+4)}$

* Bounds

$g(x) = \frac{4x^5}{4} - \frac{2x^3}{4} + \frac{2x^2}{4} + \frac{1}{4}$

* since L.C is not 1 factor out L.C

$4(x^5 - \frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{1}{4})$

*$\max\{1, |\frac{1}{4}| + |\frac{1}{2}| + |\frac{1}{2}|\}$

$\max\{1, \frac{5}{4}\} = \boxed{5/4}$

or

* $1 + \max\{|\frac{1}{4}|, |\frac{1}{2}|, |\frac{1}{2}|\}$

$1 + \frac{1}{2} = \boxed{\frac{3}{2}}$

since the smaller one is $\frac{5}{4}$

* Since $\frac{5}{4}$ is smaller than $\frac{3}{2}$ The bound is 5/4 which means all real zeros will lie between $-\frac{5}{4}$ to $\frac{5}{4}$

example

$f(x) = x^4 - 3x^2 - 4$

* L.C is 1

- $\max\{1, |-4| + |-3| + |1|\}$

$\{1, 7\} = \boxed{7}$

or

- $1 + \max\{|-4|, |-3|, |1|\}$

$= 1 + 4 = \boxed{5}$

The bound on f(x) is 5 which means all real zeros lie within the domain -5 to 5