8.2 Systems of Linear Equations: Matrices

System of equations vs Augmented Matrix

$$\begin{cases} x + 4y = 14 \\ 3x - 2y = 0 \end{cases} \qquad \left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

A **matrix** is defined as a rectangular

array of numbers *Think Battleship

$$\begin{array}{c|c} & \text{Column 1} \quad \text{Column 2} \quad \cdots \quad \text{Column } j \quad \cdots \quad \text{Column } n \\ \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row } i \\ \vdots \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

Write the following System of Equations into an Augmented Matrix:

(a) $\begin{cases} 3x - 2y = 3 \\ -2x + y = -2 \end{cases} = \left[\begin{array}{cc|c} 3 & -2 & 3 \\ -2 & 1 & -2 \end{array}\right]$

(b) $\begin{cases} 3x - 2y + 5z = 0 \\ -2x + 4z + 2 = 0 \\ x + 4y - 7 = 0 \end{cases}$ → get in order

$3x - 2y + 5z = 0$
$-2x + 0 + 4z = 2$
$x + 4y + 0 = 7$

$$= \left[\begin{array}{ccc|c} 3 & -2 & 5 & 0 \\ -2 & 0 & 4 & -2 \\ 1 & 4 & 0 & 7 \end{array}\right]$$

Write the following Augmented Matrix into a System of Equations:

(a) $\left[\begin{array}{cc|c} -2 & 1 & 3 \\ 1 & 1 & -2 \end{array}\right] = \begin{array}{l} -2x + y = 3 \\ x + y = -2 \end{array}$

(b) $\left[\begin{array}{ccc|c} 3 & -2 & 5 & 3 \\ 3 & -2 & 1 & -2 \\ 4 & -2 & 1 & 1 \end{array}\right] = \begin{array}{l} 3x - 2y + 5z = 3 \\ 3x - 2y + z = -2 \\ 4x - 2y + z = 1 \end{array}$

Row Operations Rules:

1. Interchange any two rows.
2. Replace a row by a nonzero multiple of that row
3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

Augmented Matrix →

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

Row Echelon →

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Reduced Row Echelon

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

How to apply Row Operations:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

$R_2 = -3r_1 + r_2$

↳ only R_2 is changing

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1)+3 & -3(-2)-5 & -3(2)+9 \end{array}\right]$$

$$= \boxed{\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]}$$

Find the operation necessary to result in the augmented matrix having a 0 in row 1 column 2.

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$

what do I need to do to make that -2 a 0?

$R_1 = 2(R_2) + R_1$

$$\left[\begin{array}{cc|c} 2(0)+1 & 2(1)-2 & 2(3)+2 \\ 0 & 1 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 3 \end{array}\right]$$

Solve the following Matrix by finding the REF (Row Echelon Form)

$$\begin{array}{c} \quad x \qquad y \qquad z \quad \text{constants} \end{array}$$

Solve: $\begin{cases} 2x + 2y \phantom{{}+z} = 6 & (1) \\ x + y + z = 1 & (2) \\ 3x + 4y - z = 13 & (3) \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$ need a_{11}

$\xrightarrow{\text{switch } R_1 \,\&\, R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$ need $a_{21}=0$

$\xrightarrow{R_2 = -2(R_1)+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -2(1)+2 & -2(1)+2 & -2(1)+0 & -2(1)+6 \\ 3 & 4 & -1 & 13 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 3 & 4 & -1 & 13 \end{array}\right]$ need $a_{31}=0$

$\xrightarrow{R_3 = -3(R_1)+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$ need $a_{22}=1$

$\xrightarrow{\text{switch Rows } R_2 \; R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$ need $a_{33}=1$

it also made $a_{32}=0$ yah!

$\xrightarrow{R_3 = -\frac{1}{2}(R_3)} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$

Thus $x+y+z=1 \Rightarrow x+2-2=1$ $\boxed{x=1}$

Thus $y-4z=10 \Rightarrow y-4(-2)=10 \Rightarrow \boxed{y=2}$

Thus $\boxed{z=-2}$

now use substitution

$\boxed{(1,2,-2)}$ interval notation

$\boxed{\{x,y,z \mid x=1,\ y=2,\ z=-2\}}$ set notation

$$\begin{array}{c} \quad x \qquad y \qquad z \quad \text{constants} \end{array}$$

Solve: $\begin{cases} x - y + z = 8 \\ 2x + 3y - z = -2 \\ 3x - 2y - 9z = 9 \end{cases}$

$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$ need $a_{21}=0$

$\xrightarrow{R_2 = -2(R_1)+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{array}\right]$ need $R_{31}=0$

$\xrightarrow{R_3 = -3R_1+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$ need $a_{22}=1$

$\xrightarrow{\text{switch } R_2 \,\&\, R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right]$ need $a_{31}=0$

$\xrightarrow{R_3 = -5R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$ need $a_{33}=1$

$\xrightarrow{R_3 = \frac{1}{57}R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{cases} x - y + z = 8 \\ y - 12z = -15 \\ \boxed{z=1} \end{cases}$

Now use substitution

$y = 12(1)-15$ $\boxed{y=-3}$

$x = -3-1+8 \Rightarrow \boxed{x=4}$

$\boxed{(4,-3,1)}$ interval notation

$\{x,y,z \mid x=4,\ y=-3,\ z=1\}$ set builder notation

$$\begin{array}{ccccc} & x & y & z & \text{constant} \end{array}$$

Solve: $\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases} \Rightarrow$

need $a_{11}=1$

$$\left[\begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_1 = R_1 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

need $a_{21}=0$

$$\xrightarrow{R_2 = 12(R_1) + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

need $a_{31}=0$

$$\xrightarrow{R_3 = -5R_1 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

need $a_{22}=1$

$$R_2 = \frac{-1}{22} R_2 \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 11 & -1 & -2 \end{array}\right]$$

need $a_{32}=0$

$$\xrightarrow{R_3 = -11R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x - 2y = 1 \rightarrow x - 2\left(\frac{1}{11}z - \frac{2}{11}\right) = 1 \Rightarrow x = \frac{2}{11}z + \frac{7}{11} \\ y - \frac{1}{11}z = -\frac{2}{11} \rightarrow y = \frac{1}{11}z - \frac{2}{11} \\ z = \mathbb{R} \text{ (all reals)} \end{cases}$$

since $0 = 0$ $z = \mathbb{R}$

$$\boxed{\left\{ x, y, z \mid x = \frac{2}{11}z + \frac{7}{11},\ y = \frac{1}{11}z - \frac{2}{11},\ z = \mathbb{R} \right\}}$$

Solve: $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases} \Rightarrow$

need $a_{21}=0$ & $a_{31}=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = -1R_1 + R_3]{R_2 = -2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right]$$

need $a_{22}=1$

Swith R_2 & R_3

need $a_{32}=0$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array}\right] \xrightarrow{R_3 = 3R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right]$$

$0 \neq 27$ Thus the system is $\boxed{\text{inconsistent}}$

$\emptyset$ or $\{ \}$

means the empty set

Solve: $\begin{cases} x - 2y + z = 0 \\ 2x + 2y - 3z = -3 \\ y - z = -1 \\ -x + 4y + 2z = 13 \end{cases}$

need $a_{21}=0$ & $a_{41}=0$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & -3 \\ 0 & 1 & -1 & -1 \\ -1 & 4 & 2 & 13 \end{array}\right] \xrightarrow[R_4 = R_1 + R_4]{R_2 = 2R_1 + R_2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -5 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 3 & 13 \end{array}\right]$$

need $a_{22}=1$

Switch Rows R_2 & R_3

need a_{32} & a_{42}

$$\xrightarrow{} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -5 & -3 \\ 0 & 2 & 3 & 13 \end{array}\right] \xrightarrow[R_4 = -2R_2 + R_4]{R_3 = -6R_2 + R_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

need $a_{43}=0$

need $a_{12}=0$

$$\xrightarrow{R_4 = -5R_3 + R_4} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 = 2R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

need a_{13} & $a_{23}=0$

$$\xrightarrow[R_2 = R_3 + R_2]{R_1 = R_3 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

since $0=0$ the ~~a~~ system is consistent so now to now find the 3 variables keep going to get matrix in Reduced Row Echelon

$\boxed{\{x, y, z \mid x=1, y=2, z=3\}}$ or $\boxed{(1,2,3)}$

Mckaylie and Jordan require an additional $25,000 in annual income (to supplement their music addiction). The risk assessment determined the best options for their investments were Treasury notes that yield 3%, Treasury bonds that yield 5%, or Corporate bonds that yield 6%. If they have $600,000 to invest and want the amount invested in Treasury notes to equal the amount invested in Treasury bonds, and Corporate bonds. How much should be placed in each investment?

let a = \$ in Treasury notes
b = \$ in Treasury bonds
c = \$ in Corporate bonds

Thus $a + b + c = \$600{,}000$

$.03a + .05b + .06c = \$25{,}000$

$a = b + c$ or $a - b - c = 0$

Since we have 3 variables & 3 equations we can now solve

need $a_{21}=0$, & $a_{31}=0$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ .03 & .05 & .06 & 25{,}000 \\ 1 & -1 & -1 & 0 \end{array}\right] \xrightarrow[R_3 = -R_1 + R_2]{R_2 = -.03R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600{,}000 \\ 0 & .02 & .03 & 7{,}000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right]$$

need $a_{22}=1$

$$\xrightarrow{R_2 = \frac{1}{.02}R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600000 \\ 0 & 1 & 1.5 & 350{,}000 \\ 0 & -2 & -2 & -600{,}000 \end{array}\right]$$

need $a_{31}=0$

$$\xrightarrow{R_3 = 2R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 600000 \\ 0 & 1 & 1.5 & 350000 \\ 0 & 0 & 1 & 100{,}000 \end{array}\right] \Rightarrow \begin{cases} a + b + c = 600{,}000 \\ b + 1.5c = 350{,}000 \\ \boxed{c = 100{,}000} \end{cases}$$

$b + 1.5(100{,}000) = 350{,}000$

$\boxed{b = 200{,}000}$

$a + 200{,}000 + 100{,}000 =$
$a + 300{,}000 = 600{,}000$

$\boxed{a = 300{,}000}$

So Mckaylie & Jordan should invest:
\$300,000 in Treasury notes
\$200,000 in Treasury bonds
\$100,000 in Corporate bonds