8.4 Matrix Algebra

Adding and Subtracting Matrices

If A and B are both $m \times n$ matrices then the **sum** of A and B, denoted $A + B$, is a matrix obtained by adding ***corresponding entries*** of A and B. The **difference** of A and B, denoted $A - B$, is obtained by subtracting ***corresponding entries*** of A and B.

Commutative Property of Matrix Addition

$$A + B = B + A$$

Associative Property of Matrix Addition

$$(A + B) + C = A + (B + C)$$

Zero Matrix

$$A + 0 = 0 + A = A$$

Scalar Multiples of a Martix

If A is an $m \times n$ matrix and s is a scalar, then we let kA denote the matrix obtained by multiplying every element of A by k. This procedure is called **scalar multiplication.**

Properties of Scalar Multiplication

$$k(hA) = (kh)A$$
$$(k + h)A = kA + hA$$
$$k(A + B) = kA + kB$$

Example: Find a) A + B b) A - B

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -4 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+-3, & -2+0, & 2+4 \\ 0+2, & -1+1, & 3+-4 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 6 \\ 2 & 0 & -1 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 1--3, & -2-0, & 2-4 \\ 0-2, & -1-1, & 3--4 \end{bmatrix}$$

$$A-B = \begin{bmatrix} 4 & -2 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Example: Find a) 4A b) 3A - 2B

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & -1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

a) $4A = \begin{bmatrix} 4 & -8 & 8 \\ 0 & -4 & 12 \end{bmatrix}$

b) 3A-2B

$$3A = \begin{bmatrix} 3 & -6 & 6 \\ 0 & -3 & 9 \end{bmatrix} \quad 2B = \begin{bmatrix} 0 & -4 & 6 \\ 2 & 4 & 2 \end{bmatrix}$$

$$3A-2B = \begin{bmatrix} 3-0 & -6--4 & 6-6 \\ 0-2 & -3-4 & 9-2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -2 & -7 & 7 \end{bmatrix}$$

Always multiply the row 1 by column 1 then row 1 by column 2 etc.

The Product of Matrices

The **product** RC of R times C is defined as the number

$$\boxed{RC = [r_1 \quad r_2 \cdots r_n]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n}$$

must have same # of elements

A: m by r B: r by n

Must be same for AB to be defined

AB is m by n.

Solution is $m \times n$ matrix

Example: Find the product of the row vector and column vector $[1\times 3] \times [3\times 1]$ (same)

Solution is $[1\times 1]$

Find RC if $R = [1 \quad -2 \quad 4]$ and $C = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$RC = 1(2) + -2(-1) + 4(3)$

$RC = 2+2+12$

$RC = \boxed{16}$

Matrix Multiplication is NOT Commutative **!!!!**

Associative Property of Matrix Multiplication

$$\boxed{A(BC) = (AB)C}$$

Distributive Property

$$\boxed{A(B + C) = AB + AC}$$

Find the product AB if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Remember Rows first by each column then the next row by each column

$$AB = \begin{bmatrix} 6+2-3 & 12-6+1 \\ 0\ -4+3 & 0\ 12\ -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 7 \\ -1 & 11 \end{bmatrix}$$

2×2

Remember dimensions?

$A = 2\times 3$ $B = 3\times 2$

so

$2\times 3 \times 3\times 2$

Row / Have to be the same / column

Solution is 2×2 matrix

Find the product BA if

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

* since $A = 2\times 3$ $B = 3\times 2$

$BA = 3\times 2 \times 2\times 3$

3×3 for my solution

row first

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 6+0 & -4+16 & 2-4 \\ -3+0 & 2+12 & -1-3 \\ -9+0 & 6+4 & -3-1 \end{bmatrix} = \begin{bmatrix} 6 & 12 & -2 \\ -3 & 14 & -4 \\ -9 & 10 & -4 \end{bmatrix} = BA$$

notice its a 3×3

* notice from our example that $AB \neq BA$ so Matrix multiplication is NOT COMMUTATIVE

Identity Matrices

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Identity Property

If A is an m by n matrix, then

$$\boxed{I_m A = A \quad \text{and} \quad AI_n = A}$$

If A is an n by n square matrix, then

$$\boxed{AI_n = I_n A = A}$$

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

* if $AX = B$ Then

$A^{-1}B = X$ and

$BA^{-1} = X$

Example: Multiplication with Identity Matrices

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

Find: (a) AI_3 $\quad 2 \times 3 \ \text{X} \ 3 \times 3$

$$AI_3 = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3+0+0, & 0+-2+0, & 0+0+1 \\ 0+0+0 & 0+4+0 & 0+0+-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$ Thus $AI_3 = A$

(b) $I_2 A$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 3+0, & -2+0, & 1+0 \\ 0+0, & 0+4 & 0-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$ Thus $I_2 A = A$

Show that the inverse of $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$.

if $AA^{-1} = I$ then they are inverses

$$AA^{-1} = \begin{bmatrix} 3-2 & , +\frac{3}{2} - \frac{3}{2} \\ -4+4 & , -2+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus A is an inverse to A^{-1} and A^{-1} is an inverse to A

Procedure for Finding the Inverse of a Nonsingular Matrix

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

STEP 1: Form the matrix $[A|I_n]$.

STEP 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

STEP 3: The reduced row echelon form of $[A|I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.

The matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ is nonsingular. Find its inverse.

$[A|I_3]$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = -2R_1 + R_2$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -3 & -2 & 0 & 1 \end{array}\right]$ $R_2 = -R_2$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 4 & -3 & -2 & 0 & 1 \end{array}\right]$ $R_3 = -4R_2 + R_3$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 9 & -2 & 4 & 1 \end{array}\right]$

$R_3 = \frac{1}{9}R_3$ $\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{array}\right]$ $R_1 = R_2 + R_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{array}\right]$ $R_1 = R_3 + R_1$, $R_2 = 3R_3 + R_2$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{array}\right]$

$$A^{-1} = \begin{bmatrix} \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{bmatrix}$$

Show that the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has no inverse.

$AI_2 = \left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ 4 & -2 & 0 & 1 \end{array}\right]$ $R_1 = -\frac{1}{2}R_1$ $\left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 4 & -2 & 0 & 1 \end{array}\right]$ $R_2 = -4R_1 + R_2$ $\left[\begin{array}{cc|cc} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$

can't make the 0 a 1

since we can't get the idenity on the left side of the vertical bar A doesn't have an inverse

Solve the system of equations: $\begin{cases} x - y + 2z = 1 \\ \quad -y + 3z = -2 \\ 2x + 2y + z = -1 \end{cases}$ let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ & the solution matrix $X = [\;]$

since $AX = B$ then we know $X = A^{-1}B$ or BA^{-1} so since we know A^{-1} from the example at the top of this page

$$A^{-1}B = \begin{bmatrix} \frac{7}{9} & -\frac{5}{9} & \frac{1}{9} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{9} & \frac{4}{9} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{9} + \frac{10}{9} + \frac{1}{9} \\ -\frac{2}{3} - \frac{2}{3} + \frac{1}{3} \\ -\frac{2}{9} - \frac{8}{9} + \frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{16}{9} \\ -\frac{5}{3} \\ -\frac{11}{9} \end{bmatrix}$$

so $x = \frac{16}{9}$ $y = -\frac{5}{3}$ $z = -\frac{11}{9}$

$\left(\frac{16}{9}, -\frac{5}{3}, \frac{-11}{3}\right)$